Académie de Toulouse – classes européennes mathématique-anglais – session 2013

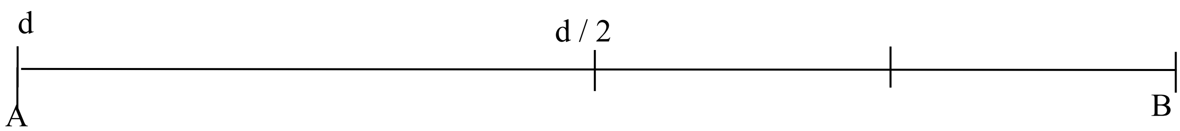
Sujet n°22

Please do not write on the exam paper.
Please do not forget to give it back at the end of the examination.
The dichotomy paradox

The following paradox has been attributed to Zeno of Elea, a greek philosopher :
"There is no motion, because that which is moved must arrive at the middle before it arrives at the end, and so on ad infinitum."

Let us consider an arrow leaving a bow at point A and going to point B at constant speed S. the distance between A and B is denoted by d.

We define an infinite sequence of steps : at each step, the arrow makes half of the remaining distance to point B. We denote by r_n the remaining distance between the arrow and point B.



A)

1. Compute r_1 , r_2 , r_3 . Show that $r_{n+1}=\frac{r_n}{2}$ for all integer n.
2. What kind of sequence is (r_n) ? Write r_n in terms of n.
3. Compute the limit of r_n as n tends to infinity .

B) Now we denote by s_n the distance made by the arrow at step n.

1. Compute s_1 and s_2 and then show that $s_{n+1}=\frac{s_n}{2}$ for all n.
2. What kind of sequence is (s_n) ? Write s_n in terms of n.

C) We denote by d_n the total distance made by the arrow from step 1 until step n.

1. Write d_1 in terms of s_1 . Write d_2 in terms of s_1 and s_2 .
2. Write d_n in terms of $s_1, s_2, \ldots, s_n$.
3. Show that $d_n=\left(\frac{d}{2}\right)\left(1+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2+\ldots+\left(\frac{1}{2}\right)^{n-1}\right)$.

(We recall the formula : $1+q+q^2+\ldots+q^{n-1}=\frac{1-q^n}{1-q}$)

4. Compute d_n in terms of n.
5. Compute the limit of d_n as n tends to infinity .